

# Curvature integral estimates for complete hypersurfaces

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*Dedicate to Professor Manfredo do Carmo on the occasion of his 80th birthday.*

**Abstract.** We consider the integrals of  $r$ -mean curvatures  $S_r$  of a complete hypersurface  $M$  in space forms  $\mathcal{Q}_c^{n+1}$  which generalize volume ( $r = 0$ ), total mean curvature ( $r = 1$ ), total scalar curvature ( $r = 2$ ) and total curvature ( $r = n$ ). Among other results we prove that a complete properly immersed hypersurface of a space form with  $S_r \geq 0$ ,  $S_r \not\equiv 0$  and  $S_{r+1} \equiv 0$  for some  $r \leq n - 1$  has  $\int_M S_r dM = \infty$ .

*Key words:* mean curvature, space form, hypersurface, volume estimate.

## 1 Introduction

Let  $M^n$  be a complete orientable immersed hypersurface of a space form  $\mathcal{Q}_c^{n+1}$  of constant sectional curvature  $c$ . Let  $A$  be the second fundamental operator of the immersion and let  $\lambda_1, \dots, \lambda_n$  be the eigenvalues of  $A$ . We define the  $r$ -mean curvature of the immersion at a point  $p$  by

$$H_r(p) = \frac{1}{\binom{n}{r}} \sum_{i_1 < \dots < i_r} \lambda_{i_1} \dots \lambda_{i_r} = \frac{1}{\binom{n}{r}} S_r(p),$$

where  $S_r$  is the  $r$  symmetric function of the  $\lambda_1, \dots, \lambda_n$ , for  $1 \leq r \leq n$ , and define  $H_0 = 1$  and  $H_r = 0$ , for all  $r \geq n + 1$ . In particular, for  $r = 1$ ,  $H_1 = H$  is the mean curvature of the immersion.

We define the  $r$ -area of a domain  $D \subset M$  by

$$\mathcal{A}_r(D) = \int_D S_r(P) dM.$$

Then, when  $r = 0$ ,  $\mathcal{A}_0$  is the volume of  $D$ .

In this paper we are interested in  $r$ -areas estimates. When  $r = 0$ , it is well known that a complete properly immersed minimal hypersurface in  $\mathbb{R}^{n+1}$  has at least polynomial volume growth. In fact volume infinity results holds for more general ambient spaces. Precisely we

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\*The authors were partially supported by CNPq and FAPERJ, Brazil.

have the following result of K. Frensel [Fr].

**Theorem** ([Fr] Theorem 1) *Let  $M^m$  be a complete, noncompact manifold and let  $x : M^m \rightarrow N^n$  be an isometric immersion with mean curvature vector field bounded in norm. If  $N^n$  has sectional curvature bounded from above and injective radius bounded from below by a positive constant, then the volume of  $M^m$  is infinite.*

It is also true that each end of  $M$  has infinite volume under the same conditions (see [CCZ]). These estimates have been used in studying the topology and geometric properties of minimal hypersurfaces and hypersurfaces with constant mean curvature (see for example [Fr], [CCZ], [Si]). It would be natural to ask the following.

**Question** *Let  $M^n$  be a complete noncompact manifold and let  $x : M^n \rightarrow N^{n+1}$  be an isometric immersion such that there is a positive constant  $C$  satisfying*

$$|S_{r+1}| \leq CS_r,$$

*for some  $r = 0, 1, \dots, n-1$ . Is the  $r$ -area of  $M^m$  infinite?*

When  $r = n$   $S_{r+1} = 0$  one can find a negative answer to this question by taking an example that  $M$  is a complete noncompact surface in  $\mathbb{R}^3$  with positive Gaussian curvature and the total curvature is finite by the theorem of Cohn-Vossen. When  $r < n$  we obtain  $r$ -area estimate and give positive answers to the question in some interesting cases.

To state our results we introduce the  $r$ 'th Newton transformation,  $P_r : T_p M \rightarrow T_p M$ , which are defined inductively by

$$\begin{aligned} P_0 &= I, \\ P_r &= S_r I - A \circ P_{r-1}, \quad r > 1. \end{aligned}$$

**Theorem A.** (Theorem 2.1) *Let  $\mathcal{Q}_c^{n+1}$  be a Riemannian manifold with constant sectional curvature  $c$  and  $M^n$  a complete noncompact properly immersed hypersurface of  $\mathcal{Q}_c^{n+1}$ . Assume that there exists a nonnegative constant  $\alpha$  such that*

$$(r+1)|S_{r+1}| \leq (n-r)\alpha S_r,$$

*for some  $r \leq n-1$ . If  $P_r$  is positive semi definite, then for any  $q \in M$  such that  $S_r(q) \neq 0$  and any  $\mu_0 > 0$  there exists a positive constant  $C$  depending on  $\mu_0, q$  and  $M$  such that*

$$\int_{\overline{B}_{\mu}(q) \cap M} S_r dM \geq \int_{\mu_0}^{\mu} C e^{-\alpha \tau} d\tau.$$

*For the case  $c > 0$ , one needs that  $\mu \leq \frac{\pi}{2\kappa}$ .*

As a consequence of this result we obtain the following, which is one of the main results of this article.

**Theorem B.** (Corollary 2.2) *Let  $\mathcal{Q}_c^{n+1}$  be a Riemannian manifold with constant sectional*

curvature  $c \leq 0$  and  $M^n$  a complete noncompact properly immersed hypersurface of  $\mathcal{Q}_c^{n+1}$ . Assume that  $S_r \geq 0$ ,  $S_r \not\equiv 0$  and  $S_{r+1} \equiv 0$  for some  $r \leq n-1$ . Then  $\int_M S_r dM = \infty$ .

**Remark 1.1** The cases when  $r$  is even and  $r$  is odd are different. If  $r$  is odd and  $S_r \leq 0$ , we can change the orientation so that  $S_r \geq 0$ . But when  $r$  is even,  $S_r$  is independent of the choice of orientation. It has been proved by Gromov and Lawson that the existence of a complete metric with nonpositive scalar curvature ( $r = 2$ ) implies some topological obstructions, which is called enlargeable (see Corollary A in [GL]). Enlargeable manifolds cannot carry metrics of positive scalar curvature.

**Remark 1.2** This result has been used to study the stable hypersurfaces with constant scalar curvature in Euclidean spaces in [ASZ].

It is known that the volume estimate of submanifold is related to the validity of Sobolev inequality. Topping [To] used Sobolev inequality to get a diameter estimate in terms of the integral of mean curvature. In the section 4, we get a global estimate of the integral of mean curvature which is sharp for cylinders. Precisely we prove that

**Theorem C.** (Theorem 4.1) *Let  $M^m$  be an  $m$ -dimensional complete noncompact Riemannian manifold isometrically immersed in  $\mathbb{R}^n$ . Then there exists a positive constant  $\delta$  depending on  $m$  such that if*

$$\limsup_{r \rightarrow +\infty} \frac{V(x, r)}{r^m} < \delta,$$

where  $V(x, r)$  denotes the volume of the geodesic ball  $B_r(x)$ , then

$$\limsup_{R \rightarrow +\infty} \frac{\int_{B_R(x)} |H|^{m-1} dM}{R} > 0.$$

In particular,  $\int_M |H|^{m-1} dM = +\infty$ .

For a complete noncompact surface  $M$  with finite total curvature, Cohn-Vossen theorem says that (see Theorem 6 in [CV])

$$\int_M K dM \leq 2\pi\chi(M)$$

A special case of Corollary 4.1 says that  $\int_M |H| dM = +\infty$  if equality holds.

The rest of the paper is organized as follows. In Section 2 we compute some formulas for distance function and  $r$ -mean curvature and apply them to main results. The estimate obtained in Section 2 can be improved when  $r = 0$  and this is demonstrated in 3. In Section 4 we give the proof of Theorem C.

**Acknowledgement.** The authors would like to thank Professor M. P. do Carmo for many invaluable comments, suggestions and encouragements. We would also like to thank M. Dajczer and L. Florit for interests and comments.

## 2 $r$ -area estimate

Let  $x : M^n \rightarrow N^{n+1}$  be an isometric immersion of a Riemannian manifold into a Riemannian manifold  $N$ .

In [Re], Reilly showed that the  $P_r$  satisfy the following

**Proposition 2.1** ([Re], p.224, see also [BC]- lemma 2.1) *Let  $x : M^n \rightarrow N^{n+1}$  be an isometric immersion between two Riemannian manifolds and let  $A$  be the second fundamental form of  $x$ . The  $r$ 'th Newton transformation  $P_r$  associated to  $A$  satisfies:*

$$\text{trace}(P_r) = (n - r)S_r, \quad (1)$$

$$\text{trace}(A \circ P_r) = (r + 1)S_{r+1}. \quad (2)$$

For hypersurfaces with bounded mean curvature, the Laplacian of the square of the intrinsic distance to a fixed point of  $M$  played an important role in the proof of Frensel's estimate of the volume of  $M$ . In the case of  $H_r$  bounded, with  $r > 1$ , we used another second order differential operator defined on  $M$ , which seems to be natural for this problem. Associated to each Newton transformation  $P_r$ , if  $f : M \rightarrow \mathbb{R}$  is a smooth function, we define

$$L_r(f) = \text{trace}(P_r \circ \text{Hess } f).$$

These operators are, in a certain sense, a generalization of the Laplace operator since  $L_0(f) = \text{trace}(\text{Hess } f) = \Delta f$ . They were introduced by Voss [Vo] in connection with variational arguments. In general, these operators are not elliptic and some conditions are necessary to ensure the presence of ellipticity.

We include here some useful facts.

**Proposition 2.2** ([El]- Lemma 3.10) *Let  $N^{n+1}$  be an  $(n+1)$ -dimensional oriented Riemannian manifold and let  $M^n$  be a connected  $n$ -dimensional orientable Riemannian manifold. Suppose  $x : M \rightarrow N$  is an isometric immersion. If  $H_2 > 0$ , then the operator  $L_1$  is elliptic.*

**Proposition 2.3** ([CR]- Proposition 3.2) *Let  $N^{n+1}$  be an  $(n+1)$ -dimensional oriented Riemannian manifold and let  $M^n$  be a connected  $n$ -dimensional orientable Riemannian manifold (with or without boundary). Suppose  $x : M \rightarrow N$  is an isometric immersion with  $H_r > 0$  for some  $1 \leq r \leq n$ . If there exists an interior point  $p$  of  $M$  such that all the principle curvatures at  $p$  are nonnegative, then for all  $1 \leq j \leq r - 1$ , the operator  $L_j$  is elliptic, and the  $j$ -mean curvature  $H_j$  is positive.*

We need the following proposition which is essentially the content of Lemma 1.1 and equation (1.3) of [HL]. We include here with a direct proof.

**Proposition 2.4** *Let  $M^n \rightarrow N^{n+1}$  be an isometric immersion. Suppose that  $S_{r+1}(p) = 0$ , for some  $r$ ,  $0 \leq r < n$ . Then  $P_r$  is semi definite at  $p$ .*

**Proof.** Consider  $S_r = S_r(\lambda_1, \dots, \lambda_n)$ . Then  $\frac{\partial S_r}{\partial \lambda_i}$  are the eigenvalues of  $P_r$ . Let  $(\lambda_1^0, \dots, \lambda_n^0)$  be the principal curvatures of  $M$  at  $p$ . Hence

$$S_{r+1}(\lambda_1^0, \dots, \lambda_n^0) = 0.$$

We choose  $\epsilon = \min_{\lambda_i^0 \neq 0} \{1, |\lambda_i^0|\}$ . Then, for all  $(\varepsilon_1, \dots, \varepsilon_n)$  with  $\varepsilon_i \in (0, \epsilon)$ ,  $S_{r+1}(\lambda_1^0 + \varepsilon_1, \dots, \lambda_n^0 + \varepsilon_n)$

does not change sign. This implies that  $\frac{\partial S_r}{\partial \lambda_i} \geq 0$  for all  $i = 1, \dots, n$  or  $\frac{\partial S_r}{\partial \lambda_i} \leq 0$  for all  $i = 1, \dots, n$ . Thus  $P_r$  is semi definite at  $p$ . □

Let  $M^n$  and  $N^{n+1}$  be Riemannian manifolds and  $x : M^n \rightarrow N^{n+1}$  an isometric immersion. Henceforth, we shall tacitly make the usual identification of  $X \in T_p M$  with  $dx_p(X)$ . In particular, if  $F : N \rightarrow \mathbb{R}$  is smooth and we consider the composition  $f = F \circ x$ , then we have at  $p \in M$ , for every  $X \in T_p M$ :

$$\langle \text{grad}_M f, X \rangle = df(X) = dF(X) = \langle \text{grad}_N F, X \rangle,$$

where  $\text{grad}_M$  and  $\text{grad}_N$  denotes the gradient on  $M$  and the gradient on  $N$ , respectively. So that

$$\text{grad}_N F = \text{grad}_M f + (\text{grad}_N F)^\perp, \quad (3)$$

where  $(\text{grad}_N F)^\perp$  is perpendicular to  $T_p M$ . Let  $F : N \rightarrow \mathbb{R}$  be a  $C^2$  function and denote  $f : M \rightarrow \mathbb{R}$  the function induced by  $F$  by restriction, that is  $f = F \circ x$ . We have the following.

**Lemma 2.1** *Let  $x : M^n \rightarrow N^{n+1}$  an isometric immersion. Let  $F : N \rightarrow \mathbb{R}$  a smooth function and consider  $f = F \circ x : M \rightarrow \mathbb{R}$ . For an orthonormal frame  $\{e_i\}$  on  $M$ , we have*

$$L_r f = \sum_{i=1}^n \text{Hess}(F)(e_i, P_r(e_i)) + (r+1)S_{r+1} \langle \text{grad}_N F, \eta \rangle, \quad (4)$$

where  $\eta$  denotes the normal vector field of the immersion and  $\text{grad}_N$  is the gradient of  $N$ .

**Proof.** Let  $\nabla$  and  $\bar{\nabla}$  the connection of  $M$  and  $N$ , respectively. If  $\alpha$  denotes the second fundamental form of the immersion, Gauss' equation and equations (2) and (3) imply that

$$\begin{aligned} L_r f &= \sum_{i=1}^n \langle \nabla_{e_i}(\text{grad}_M f), P_r(e_i) \rangle \\ &= \sum_{i=1}^n \langle \bar{\nabla}_{e_i}(\text{grad}_M f) - [\bar{\nabla}_{e_i}(\text{grad}_M f) - \nabla_{e_i}(\text{grad}_M f)], P_r(e_i) \rangle \\ &= \sum_{i=1}^n \langle \bar{\nabla}_{e_i}(\text{grad}_M f) - \alpha(e_i, \text{grad}_M f), P_r(e_i) \rangle \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^n \langle \bar{\nabla}_{e_i}(\text{grad}_M f), P_r(e_i) \rangle \\
&= \sum_{i=1}^n \langle \bar{\nabla}_{e_i}(\text{grad}_N F - (\text{grad}_N F)^\perp), P_r(e_i) \rangle \\
&= \sum_{i=1}^n \langle \bar{\nabla}_{e_i} \text{grad}_N F, P_r(e_i) \rangle - \sum_{i=1}^n \langle \bar{\nabla}_{e_i}(\text{grad}_N F)^\perp, P_r(e_i) \rangle \\
&= \sum_{i=1}^n \text{Hess}(F)(e_i, P_r(e_i)) - \sum_{i=1}^n \langle \bar{\nabla}_{e_i}(\langle \text{grad}_N F, \eta \rangle \eta), P_r(e_i) \rangle \\
&= \sum_{i=1}^n \text{Hess}(F)(e_i, P_r(e_i)) - \sum_{i=1}^n \langle \langle \text{grad}_N F, \eta \rangle \bar{\nabla}_{e_i} \eta, P_r(e_i) \rangle \\
&= \sum_{i=1}^n \text{Hess}(F)(e_i, P_r(e_i)) - \langle \text{grad}_N F, \eta \rangle \sum_{i=1}^n \langle -A(e_i), P_r(e_i) \rangle \\
&= \sum_{i=1}^n \text{Hess}(F)(e_i, P_r(e_i)) + \langle \text{grad}_N F, \eta \rangle \sum_{i=1}^n \langle e_i, AP_r(e_i) \rangle \\
&= \sum_{i=1}^n \text{Hess}(F)(e_i, P_r(e_i)) + \langle \text{grad}_N F, \eta \rangle \text{trace}(AP_r) \\
&= \sum_{i=1}^n \text{Hess}(F)(e_i, P_r(e_i)) + (r+1)S_{r+1} \langle \text{grad}_N F, \eta \rangle.
\end{aligned}$$

□

Let  $c \in \mathbb{R}$ . Define the function  $c_\kappa(t) = \int_0^t s_\kappa(t) dt$  where

$$s_\kappa(t) = \begin{cases} \frac{\sin \kappa t}{\kappa}, & \text{if } c = \kappa^2; \\ t, & \text{if } c = 0; \\ \frac{\sinh \kappa t}{\kappa}, & \text{if } c = -\kappa^2. \end{cases} \quad (5)$$

If  $\rho$  denotes the distance function to the point  $Q$  in  $N^{n+1}$ , let  $F : N^{n+1} \rightarrow \mathbb{R}$  given by  $F(p) = c_\kappa(\rho(p))$ . Therefore the lemma 2.1 with  $f = F \circ x$ , where  $F = c_\kappa \circ \rho$  implies

**Corollary 2.1** *Let  $M$  be an immersed hypersurface in  $N^{n+1}$  and let  $\kappa \in \mathbb{R}$ . Then, for all  $p \in M$ ,*

$$L_r(c_\kappa(\rho(p))) = (n-r)s'_\kappa(\rho(p))S_r + (r+1)S_{r+1}s_\kappa(\rho(p))\langle \text{grad}_N \rho(p), \eta \rangle. \quad (6)$$

*In particular, when  $c = 0$ ,*

$$\frac{1}{2}L_r(\rho^2(p)) = (n-r)S_r + (r+1)S_{r+1}\rho(p)\langle \text{grad}_N \rho(p), \eta \rangle.$$

**Proof.** First observe that

$$\text{Hess}F(X, Y) = s_\kappa(\rho)\langle X, Y \rangle, \quad (7)$$

where  $X, Y \in T_{x(p)}\mathcal{Q}$ . In fact,

$$\begin{aligned} \text{Hess}F(X, Y) &= \text{Hess}(c_\kappa(\rho)) \\ &= \langle \bar{\nabla}_X \text{grad}_N(c_\kappa(\rho)), Y \rangle \\ &= \langle \bar{\nabla}_X s_\kappa(\rho) \text{grad}_N \rho, Y \rangle \\ &= s_\kappa(\rho) \text{Hess}\rho(X, Y) + s'_\kappa \langle \langle \text{grad}_N \rho, X \rangle \text{grad}_N \rho, Y \rangle. \end{aligned}$$

On the other hand, see [AF], p.6,

$$\text{Hess}\rho(X, Y) = \langle \bar{\nabla}_X \text{grad}_N \rho, Y \rangle = \frac{s'_\kappa(\rho)}{s_\kappa(\rho)} [\langle X, Y \rangle - \langle \text{grad}_N \rho, X \rangle \langle \text{grad}_N \rho, Y \rangle].$$

This concludes the proof of (7). Now, by using equation (3), we have

$$\begin{aligned} L_r f &= \sum_{i=1}^n s'_\kappa(\rho) \langle e_i, P_r(e_i) \rangle - (r+1) S_{r+1} \langle \text{grad}_N(c_\kappa \circ \rho), \eta \rangle \\ &= s'_\kappa(\rho) \text{trace} P_r - (r+1) S_{r+1} s_\kappa(\rho) \langle \text{grad}_N \rho, \eta \rangle. \end{aligned}$$

Finally, by using equation (1), we conclude the proof of equation (6). The case  $c = 0$  follows immediately.  $\square$

Let  $\mathcal{Q}_c^{n+1}$  be a Riemannian manifold with constant sectional curvature  $c$  and let  $x : M \rightarrow \mathcal{Q}_c^{n+1}$  an isometric immersion. It follows from Codazzi equation (see Rosenberg [Ro], p.225) that  $L_r$  is a divergent form operator, that is,

$$L_r(f) = \text{div}_M(P_r \nabla f),$$

for all smooth function  $f : M \rightarrow \mathbb{R}$ . Denote by  $B_r(Q)$  the geodesic ball of  $\mathcal{Q}_c^{n+1}$  with radius  $r$ , and center  $Q \in \mathcal{Q}_c^{n+1}$  and by  $\bar{B}_r(Q)$  its closure. We will use the following proposition to prove our results.

**Proposition 2.5** *Let  $\mathcal{Q}_c^{n+1}$  be a Riemannian manifold with constant sectional curvature  $c$  and let  $x : M^n \rightarrow \mathcal{Q}_c^{n+1}$  an isometric immersion. For  $Q \in \mathcal{Q}_c^{n+1}$ , we denote by  $\rho(x)$  the distance to the point  $Q \in \mathcal{Q}_c^{n+1}$  and  $\rho(x(p))$ ,  $p \in M$  its restriction to  $M$ . If for some  $r \leq n-1$ ,  $S_r \geq 0$ , then*

$$\int_{\partial D} s_\kappa(\rho(q)) \langle P_r(\text{grad}_{\mathcal{Q}} \rho(q)^\top), \nu \rangle dA \geq (n-r) \int_D (s'_\kappa(\rho(q)) S_r - \frac{r+1}{n-r} |S_{r+1}| s_\kappa(\rho(q))) dM, \quad (8)$$

where  $q = x(p)$ ,  $\nu$  is the conormal vector of  $D$  and  $D \subset M$  is a bounded domain with nonempty boundary  $\partial D$ . In the case  $c > 0$ , we also request that  $D \subset \bar{B}_{\frac{\pi}{2\kappa}}(Q)$ .

**Proof.** By using (6), and since  $|\text{grad}_{\mathcal{Q}}\rho(x(p))| \leq 1$ ,  $s'_\kappa(\rho(x(p))) \geq 0$ , we have

$$L_r(c_\kappa(\rho(x))) \geq (n-r)[s'_\kappa(\rho)S_r - \frac{r+1}{n-r}|S_{r+1}|s_\kappa(\rho)].$$

Integrating this inequality, we get

$$\int_D L_r(c_\kappa(\rho(x)))dM \geq (n-r) \int_D [s'_\kappa(\rho(x))S_r - \frac{r+1}{n-r}|S_{r+1}|s_\kappa(\rho(x))]dM. \quad (9)$$

On the other hand, using the divergence theorem, we have that

$$\begin{aligned} \int_D L_r(c_\kappa(\rho(x)))dM &= \int_D \text{div} P_r(\text{grad}_M(c_\kappa(\rho(x(p)))))) \\ &= \int_D \text{div}(s_\kappa \rho(x(p)) P_r(\text{grad}_{\mathcal{Q}}\rho)^\top) \\ &= \int_{\partial D} s_\kappa(\rho(x)) \langle P_r((\text{grad}_{\mathcal{Q}}\rho)^\top), \nu \rangle dA, \end{aligned}$$

where  $\nu$  denotes the outward unit normal vector field on  $\partial D$ . Therefore, if  $q = x(p)$

$$\int_{\partial D} s_\kappa(\rho(q)) \langle P_r((\text{grad}_{\mathcal{Q}}\rho(q))^\top), \nu \rangle dA \geq (n-r) \int_D [s'_\kappa(\rho(x))S_r - \frac{r+1}{n-r}|S_{r+1}|s_\kappa(\rho(x))]dM,$$

and the proposition is proved.  $\square$

Observe that the above Proposition is valid for a more general class of domains. For instance it is valid in the setting of Gauss-Green Theorem (see [Fe], p.478). In particular, if we take  $D$  to be the intersection of the extrinsic ball with  $M$  i.e.  $D = \overline{B}_\mu \cap M$  in Proposition 2.5, we have the following

**Theorem 2.1** *Let  $\mathcal{Q}_c^{n+1}$  be a Riemannian manifold with constant sectional curvature  $c$  and  $M^n$  a complete noncompact properly immersed hypersurface of  $\mathcal{Q}_c^{n+1}$ . Assume that there exists a nonnegative constant  $\alpha$  such that*

$$(r+1)|S_{r+1}| \leq (n-r)\alpha S_r, \quad (10)$$

*for some  $r \leq n-1$ . If  $P_r$  is semi-positive definite, then for any  $q \in M$  such that  $S_r(q) \neq 0$  and any  $\mu_0 > 0$  there exists a positive constant  $C$  depending on  $\mu_0$ ,  $q$  and  $M$  such that*

$$\int_{\overline{B}_\mu(q) \cap M} S_r dM \geq \int_{\mu_0}^\mu C e^{-\alpha\tau} d\tau.$$

*For the case  $c > 0$ , one needs that  $\mu \leq \frac{\pi}{2\kappa}$ .*



**Proof.** We use the notation introduced in Proposition 2.5. Take  $D = D_\tau = \overline{B}_\tau(q) \cap M$ . Since the immersion is proper, we have that  $\partial D_\tau \neq \emptyset$ , for all  $\tau > 0$ . Thus, by using (10) in equation (8), we obtain that

$$\begin{aligned} \int_{\partial D_\tau} s_\kappa(\rho(x)) \langle P_r(\text{grad}_M \rho), \nu \rangle dA &\geq (n-r) \int_{D_\tau} (s'_\kappa(\rho(x)) - \alpha s_\kappa(\rho(x))) S_r dM \\ &= (n-r) \int_0^\mu \int_{\partial D_\tau} \frac{(s'_\kappa(\rho(x)) - \alpha s_\kappa(\rho(x)))}{s_\kappa(\rho(x))} s_\kappa(\rho(x)) |\text{grad}_M \rho|^{-1} S_r dA d\tau, \end{aligned} \quad (11)$$

where we have used the co-area formula (see [Be], p. 80). Observe that the conormal vector  $\nu$  to  $\partial D$  is parallel to  $\text{grad}_M \rho$ . This fact and that  $P_r$  is semi-positive definite, implies that

$$\langle P_r(\text{grad}_M \rho), \nu \rangle \leq \text{tr}(P_r) |\text{grad}_M \rho| = (n-r) S_r |\text{grad}_M \rho|.$$

Using the above equation and the fact that along  $\partial D_\tau$ ,  $\rho(x) = \tau$ , we get

$$\int_{\partial D_\mu} s_\kappa(\rho(x)) |\text{grad}_M \rho| S_r dA \geq \int_0^\mu \frac{s'_\kappa(\tau) - \alpha s_\kappa(\tau)}{s_\kappa(\tau)} \int_{\partial D_\tau} s_\kappa(\rho(x)) |\text{grad}_M \rho|^{-1} S_r dA d\tau. \quad (12)$$

Now we define

$$\varphi(\tau) = \int_{\partial D_\tau} s_\kappa(\rho(x)) |\text{grad}_M \rho|^{-1} S_r dA.$$

The equation (12) implies

$$\varphi(\mu) \geq \int_0^\mu \frac{s'_\kappa(\tau) - \alpha s_\kappa(\tau)}{s_\kappa(\tau)} \varphi(\tau) d\tau.$$

If we write

$$\phi(\mu) = \int_0^\mu \frac{s'_\kappa(\tau) - \alpha s_\kappa(\tau)}{s_\kappa(\tau)} \varphi(\tau) d\tau,$$

we have

$$\phi'(\mu) \geq \frac{s'_\kappa(\mu) - \alpha s_\kappa(\mu)}{s_\kappa(\mu)} \phi(\mu).$$

Thus, by integrating from  $\mu_0 > 0$  to  $\mu$ , the above differential inequality arises

$$\ln \frac{\phi(\mu)}{\phi(\mu_0)} \geq \ln \left( \frac{s_\kappa(\mu)}{s_\kappa(\mu_0)} \right) - \alpha(\mu - \mu_0) = \ln \left[ \left( \frac{s_\kappa(\mu)}{s_\kappa(\mu_0)} \right) e^{-\alpha(\mu - \mu_0)} \right].$$

Hence,

$$\phi(\mu) \geq \frac{\phi(\mu_0)}{s_\kappa(\mu_0)} s_\kappa(\mu) e^{-\alpha\mu}.$$

Define

$$f(\mu) = \int_{D_\mu(q)} S_r dM.$$

Again by the co-area formula, it follows that

$$f(\mu) = \int_0^\mu \left( \int_{\partial D_\tau(q)} |\text{grad}_M \rho|^{-1} S_r dA \right) d\tau.$$

Since

$$f'(\mu) = \int_{\partial D_\mu(q)} |\text{grad}_M \rho|^{-1} S_r dA = \frac{1}{s_\kappa(\mu)} \varphi(\mu) \geq \frac{\phi(\mu_0)}{s_\kappa(\mu_0)} e^{-\alpha\mu},$$

then for  $\mu > \mu_0$ ,

$$f(\mu) \geq \int_{\mu_0}^\mu \frac{\phi(\mu_0)}{s_\kappa(\mu_0)} e^{-\alpha\tau} d\tau.$$

□

**Corollary 2.2** *Let  $\mathcal{Q}_c^{n+1}$  be a Riemannian manifold with constant sectional curvature  $c \leq 0$  and  $M^n$  a complete noncompact properly immersed hypersurface of  $\mathcal{Q}_c^{n+1}$ . Assume that  $S_r \geq 0$ ,  $S_r \not\equiv 0$  and  $S_{r+1} \equiv 0$  for some  $r \leq n-1$ . Then  $\int_M S_r dM = \infty$ .*

**Proof.** Since the immersion is proper, we have  $\partial(M \cap \overline{B}_\mu(q))$  is nonempty for all  $\mu > \mu_0$ . By using Proposition 2.4, since  $S_{r+1} = 0$ , we have that  $P_r$  is semi-definite. Now, the condition  $S_r \geq 0$  implies that  $P_r$  is positive semi-definite. Therefore, using Theorem 2.1, with  $\alpha = 0$ , for all  $\mu > \mu_0$ ,

$$\int_{\overline{B}_\mu \cap M} S_r dM \geq \int_{\mu_0}^\mu C e^{-\alpha\tau} d\tau = C(\mu - \mu_0).$$

Then

$$\int_M S_r dM = \infty.$$

□

**Remark 2.1** *When  $r$  is odd, the condition  $S_r \geq 0$  can be obtained by choosing the right orientation.*

The condition of semi-positiveness of  $P_2$  is satisfied when  $M$  is hypersurface immersed in  $\mathbb{R}^{n+1}$  with  $S_3 = 0$  (which is called 2-minimal) and  $S_2 > 0$ . In fact, in this case  $P_2$  is positive definite, since  $L_2$  is elliptic (see Proposition 2.2). So we have

**Corollary 2.3** *Let  $M^n$  be a complete 2-minimal noncompact properly immersed hypersurface of  $\mathbb{R}^{n+1}$  with nonnegative scalar curvature. Then either the scalar curvature is zero or the total scalar curvature is infinite.*

**Remark 2.2** *When  $n = 3$  the corollary can be proved using Theorem III in [HN] without the assumption that the immersion is proper. In this case,  $M^n$  has to be cylinder and the conclusion of the above Corollary follows immediately.*

**Remark 2.3** *The condition of semi-positiveness of  $P_r$  is also satisfied when  $M$  is a hypersurface in  $\mathbb{R}^{n+1}$  with nonnegative sectional or positive Ricci curvature. Indeed when  $\text{Ric}_M > 0$ , for each point in  $M$  the principal curvatures can be arranged as  $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_i < 0 < \lambda_{i+1} \leq \dots \leq \lambda_n$ . The positivity of Ricci curvature implies*

$$\text{Ric}_M(e_j) = \lambda_j \left( \sum_{k \neq j} \lambda_k \right) > 0, \quad \forall j = 1, \dots, n.$$

We can see that if  $i \neq 1$  and  $i \neq n - 1$ , and

$$\sum_{k \neq j} \lambda_k < 0, \quad \text{when } j \leq i, \quad (13)$$

$$\sum_{k \neq j} \lambda_k > 0, \quad \text{when } j > i. \quad (14)$$

From (13) we have for  $j \leq i$ ,

$$\left( \sum_{k=1}^i \lambda_k - \lambda_i \right) + \sum_{k=i+1}^n \lambda_k < 0,$$

Thus

$$-\sum_{k=1}^i \lambda_k > -\sum_{k=1}^i \lambda_k + \lambda_j > \sum_{k=i+1}^n \lambda_k.$$

On the other hand, (14), with  $j > i$ , implies that

$$\sum_{k=1}^i \lambda_k + \sum_{k=i+1}^n \lambda_k - \lambda_j > 0$$

Hence,

$$-\sum_{k=1}^i \lambda_k < -\sum_{k=1}^i \lambda_k + \lambda_j < \sum_{k=i+1}^n \lambda_k,$$

which is a contradiction. One can easily see that the cases  $i = 1$  and  $i = n - 1$  also can not occur. Thus, all  $\lambda_i$  has the same sign (we are indebted to F. Fontenele for this observation). So we can choose an orientation such that  $P_r$  is positive definite and  $S_r > 0$ .

Thus we have.

**Corollary 2.4** *Let  $M^n$  be a complete noncompact properly immersed hypersurface of  $\mathbb{R}^{n+1}$  with positive Ricci curvature. Assume that there exists a positive constant  $\alpha$  such that*

$$(r + 1)|S_{r+1}| \leq (n - r)\alpha S_r,$$

for some  $r \leq n - 1$ . Then for any  $q \in M$  and any  $\mu_0 > 0$  there exists a positive constant  $C$  depending on  $\mu_0$ ,  $Q$  and  $M$  such that

$$\int_{\bar{B}(\mu) \cap M} S_r dM \geq \int_{\mu_0}^{\mu} C e^{-\alpha \tau} d\tau,$$

where  $\bar{B}_\mu(q)$  is the geodesic ball in  $\mathbb{R}^{n+1}$  centered at  $q$ .

The following is a direct consequence of Theorem 2.1 and Proposition 2.3.

**Corollary 2.5** *Let  $M^n$  be a complete noncompact properly immersed hypersurface of  $\mathcal{Q}_c^{n+1}$ . Assume that  $S_r$  is positive and there exists a positive constant  $\alpha$  such that*

$$(r + 1)|S_{r+1}| \leq (n - r)\alpha S_r,$$

*for some  $r \leq n - 1$ . If there exists a point such that all principal curvatures are nonnegative, then for any  $q \in M$  and any  $\mu_0 > 0$  there exists a positive constant  $C$  depending on  $\mu_0$ ,  $Q$  and  $M$  such that*

$$\int_{\bar{B}(\mu) \cap M} S_r dM \geq \int_{\mu_0}^{\mu} C e^{-\alpha \tau} d\tau,$$

*where  $\bar{B}_\mu(q)$  is the geodesic ball in  $\mathcal{Q}_c^{n+1}$  centered at  $q$ . For the case  $c > 0$ , one needs that  $\mu \leq \frac{\pi}{2\kappa}$ .*

### 3 Volume estimates in general manifolds

In this section we consider  $N^{n+p}$  with sectional curvature bounded from above by a constant  $c$ . Let  $M^n$  be a submanifold isometrically immersed in  $N = N^{n+p}$ .

Let  $F : N \rightarrow \mathbb{R}$  be a  $C^2$  function and denote  $f : M \rightarrow \mathbb{R}$  the function induced by  $F$  by restriction. Essentially in the same way we prove Lemma 2.1, we obtain

$$\Delta f = \sum_{i=1}^n \text{Hess} F(e_i, e_i) + n \langle \text{grad}_N F, \mathbf{H} \rangle,$$

where  $\{e_1, e_2, \dots, e_n\}$  is an orthonormal frame along  $M$  and  $\mathbf{H}$  is the mean curvature vector. Similar to Proposition 2.5, we have

**Proposition 3.1** *Let  $N$  be a Riemannian manifold with sectional curvature bounded above by  $c$  and  $M^n$  an immersed connected submanifold of  $N$ . We denote by  $\rho(x)$  the distance between  $x$  and  $Q \in N^{n+p}$  and  $\rho(x)$  the induced function of  $\rho$  by restriction. Then*

$$\int_{\partial D} s_\kappa(\rho(x)) \langle \text{grad}_M \rho, \nu \rangle dA \geq n \int_D (s'_\kappa(\rho(x)) - |\mathbf{H}| s_\kappa(\rho(x))) dM, \quad (15)$$

where  $\kappa = \sqrt{|c|}$ ,  $\nu$  is the conormal vector of  $D$  and  $D \subset M$  is a bounded domain with nonempty boundary  $\partial D$  and  $D \cap C_N(Q) = \emptyset$ , where  $C_N(Q)$  is the cut locus of the point  $Q$  in  $N$ .

**Proof.** Let  $V = s_\kappa(\rho)\text{grad}_N\rho$  and  $V^\top$  the orthogonal projection of  $V$  into the tangent space of  $M$ . Then we have  $V^\top = s_\kappa(\rho)\text{grad}_M\rho$ , where  $\rho(x)$  is the induced function of  $\rho$  to  $M$  by restriction. Thus, Lemma 2.5 of [JK], p. 713, implies that when  $\rho < \text{inj}_N(Q)$ ,

$$\text{Hess}F(X, X) \geq s'_\kappa(\rho)\langle X, X \rangle. \quad (16)$$

Then

$$\langle \bar{\nabla}_{e_i} V, e_i \rangle \geq s'_\kappa(\rho),$$

for all  $\rho$  when  $c \leq 0$ , and  $\rho \leq \frac{\pi}{\kappa}$ , when  $c > 0$ . We have that

$$\Delta(c_\kappa(\rho(x))) \geq n[s'_\kappa(\rho) - s_\kappa(\rho)|\mathbf{H}|].$$

Integrating this inequality and applying Stokes' formula, we get

$$\int_{\partial D} s_\kappa \langle (\text{grad}_N \rho)^\top, \nu \rangle dA \geq n \int_D [s'_\kappa(\rho(x)) - s_\kappa(\rho(x))|\mathbf{H}|] dM.$$

The proposition is proved.  $\square$

Similar to Proposition 2.5, the above result is valid in a more general setting, as extrinsic geodesic balls. Using this fact, we get

**Theorem 3.1** *Let  $M$  be a Riemannian manifold isometrically immersed in a geodesic ball  $\bar{B}(O, \rho_0) \subset N^{n+p}$  with codimension  $p$ . Assume that the sectional curvature of  $N$  in  $\bar{B}(O, \rho_0)$  is bounded above by  $c$  and there exists a positive constant  $\alpha$  such that*

$$|\mathbf{H}| \leq \alpha.$$

Then

$$\text{vol}(B_\mu(Q)) \geq n\omega_n \int_0^\mu s_\kappa(s)^{n-1} e^{-n\alpha s} ds,$$

where  $\kappa = \sqrt{|c|}$ ,  $\omega_n$  is the volume of the unit ball in  $\mathbb{R}^n$  and  $B_\mu(q)$  is the intrinsic geodesic ball in  $M$  with center  $q \in M$  and radius  $\mu < \text{inj}_N(q)$ .

**Proof.** Taking  $D = B_\tau(q)$  in Proposition 3.1, then

$$\langle \text{grad}_M \rho, \nu \rangle \leq |\text{grad}_M \rho|,$$

we have

$$\begin{aligned} \int_{\partial B_\tau(q)} \frac{s_\kappa(\rho(x))}{n} |\text{grad}_M \rho| dA &\geq \int_{B_\tau(q)} (s'_\kappa(\rho(x)) - \alpha s_\kappa(\rho(x))) dM \\ &= \int_0^\tau \int_{\partial B_\tau(q)} \frac{(s'_\kappa(\rho(x)) - \alpha s_\kappa(\rho(x)))}{s_\kappa(\rho(x))} s_\kappa(\rho(x)) |\text{grad}_M \rho|^{-1} dA d\tau, \end{aligned} \quad (17)$$

where we have used the co-area formula (see [Be], p. 80). Since the intrinsic distance is not less than the extrinsic one and

$$\left(\frac{s'_\kappa}{s_\kappa}\right)' \leq 0,$$

then

$$\frac{1}{n} \int_{\partial B_\mu(q)} s_\kappa(\rho(x)) |\text{grad}_M \rho| dA \geq \int_0^\mu \frac{s'_\kappa(\tau) - \alpha s_\kappa(\tau)}{s_\kappa(\tau)} \int_{\partial B_\tau(q)} s_\kappa(\rho(x)) |\text{grad}_M \rho|^{-1} dA d\tau. \quad (18)$$

Now we define

$$\varphi(\tau) = \int_{\partial B_\tau(q)} s_\kappa(\rho(x)) |\text{grad}_M \rho|^{-1} dA.$$

Equation (18) implies

$$\frac{1}{n} \varphi(\mu) \geq \int_0^\mu \frac{s'_\kappa(\tau) - \alpha s_\kappa(\tau)}{s_\kappa(\tau)} \varphi(\tau) d\tau.$$

If we write

$$\phi(\mu) = \int_0^\mu \frac{s'_\kappa(\tau) - \alpha s_\kappa(\tau)}{s_\kappa(\tau)} \varphi(\tau) d\tau,$$

we have

$$\phi'(\mu) \geq \frac{n(s'_\kappa(\mu) - \alpha s_\kappa(\mu))}{s_\kappa(\mu)} \phi(\mu).$$

Thus, by integrating from  $\varepsilon > 0$  to  $\mu$ , with  $\mu \leq \min\{inj_N(q), \frac{\pi}{2\kappa}\}$ , when  $c > 0$ , the above differential inequality arises

$$\frac{1}{n} \ln \frac{\phi(\mu)}{\phi(\varepsilon)} \geq \ln\left(\frac{s_\kappa(\mu)}{\varepsilon}\right) - \alpha(\mu - \varepsilon) = \ln\left[\left(\frac{s_\kappa(\mu)}{\varepsilon}\right) e^{-\alpha(\mu - \varepsilon)}\right].$$

Hence,

$$\frac{\phi(\mu)}{\phi(\varepsilon)} \geq \left[\left(\frac{s_\kappa(\mu)}{\varepsilon}\right) e^{-\alpha(\mu - \varepsilon)}\right]^n. \quad (19)$$

Observe that by the mean value theorem,

$$\lim_{\varepsilon \rightarrow 0} \frac{\phi(\varepsilon)}{\varepsilon^n} = \omega_n.$$

Then

$$\phi(\mu) \geq \omega_n s_\kappa(\mu)^n e^{-n\alpha s}.$$

Now, define

$$f(\mu) = \int_{B_\mu(q)} dM = \text{vol}(B_\mu(q)).$$

Again by the co-area formula, it follows that

$$f(\mu) = \int_0^\mu \left( \int_{\partial B_\tau(q)} |\text{grad}_M \rho|^{-1} dA \right) d\tau.$$

Hence

$$f'(\mu) = \int_{\partial B_s(q)} |\text{grad}_M \rho|^{-1} dA.$$

This equality and  $|\text{grad}_M \rho| \leq 1$ , with equation (17) imply that

$$\frac{s_\kappa(\mu)}{n} f'(\mu) \geq \int_{\partial B_\tau(q)} \frac{s_\kappa(\rho(x))}{n} |\text{grad}_M \rho| dA \geq \int_0^\mu (s'_\kappa(\tau) - \alpha s_\kappa(\tau)) f'(\tau) d\tau.$$

Since

$$f'(\mu) \geq \frac{\varphi(\mu)}{s_\kappa(\mu)},$$

then

$$f(\mu) \geq \int_0^\mu \omega_n n s_\kappa(\tau)^{n-1} e^{-n\alpha\tau} d\tau,$$

which concludes the proof.  $\square$

From the theorem we have an immediate corollary.

**Corollary 3.1** (i) Let  $M^n$  an immersed minimal hypersurface of the Euclidean space  $\mathbb{R}^{n+p}$ . Then

$$\text{vol}(B_\mu(q)) \geq \omega_n \mu^n.$$

where  $\omega_n$  is the volume of the unit ball in  $\mathbb{R}^n$  and  $B_\mu(q)$  is the intrinsic geodesic ball in  $M$  with center  $q \in M$ .

(ii) Let  $M^n$  an immersed hypersurface of the hyperbolic space  $\mathbb{H}^{n+p}(-1)$ . Assume there exists a positive constant  $\alpha$  such that

$$|H| \leq \alpha < \frac{n-1}{n}.$$

Then exist a constant  $C > 0$  such that if  $\mu \geq 1$ ,

$$\text{vol}(B_\mu(q)) \geq C e^{(n-1-n\alpha)\mu},$$

where  $B_\mu(q)$  is the intrinsic geodesic ball in  $M$  with center  $q \in M$ .

## 4 Mean curvature integral

In this section, inspired by a recent work of Topping [To], we prove a type of mean curvature integral estimate for complete submanifold in a Euclidean space  $\mathbb{R}^n$  and we apply it to surfaces in  $\mathbb{R}^n$

**Theorem 4.1** Let  $M^m$  be an  $m$ -dimensional complete noncompact Riemannian manifold isometrically immersed in  $\mathbb{R}^n$ . Then there exists a positive constant  $\delta$  depending on  $m$  such that if

$$\limsup_{r \rightarrow +\infty} \frac{V(x, r)}{r^m} < \delta, \tag{20}$$

where  $V(x, r)$  denotes the volume of the geodesic ball  $B_r(x)$ , then

$$\limsup_{R \rightarrow +\infty} \frac{\int_{B_R(x)} |H|^{m-1} dM}{R} > 0. \quad (21)$$

In particular,  $\int_M |H|^{m-1} dM = +\infty$ .

We need the following lemma of Topping [To].

**Lemma 4.1** ([To], Lemma 1.2) *Let  $M^m$  be an  $m$ -dimensional complete Riemannian manifold isometrically immersed in  $\mathbb{R}^n$ . Then exists a positive constant  $\delta$  depending on  $m$  such that for any  $x \in M$  and  $R > 0$ , at least one of the following is true:*

- (i)  $\sup_{r \in (0, R]} r^{-\frac{1}{m-1}} [V(x, r)]^{-\frac{m-2}{m-1}} \int_{B(x, r)} |H|^{m-1} dM > \delta,$
- (ii)  $\inf_{r \in (0, R]} \frac{V(x, r)}{r^m} > \delta.$

**Proof of Theorem 4.1.** We can choose  $L$  large enough so that  $V(z, L) \leq \delta L^m$  for all  $z \in M$ . Then from Lemma 4.1, we have

$$\sup_{r \in (0, L]} r^{-\frac{1}{m-1}} [V(z, r)]^{-\frac{m-2}{m-1}} \int_{B_r(z)} |H|^{m-1} dM > \delta.$$

Since

$$\int_{B_r(z)} |H| dm \leq \left( \int_{B_r(z)} |H|^{m-1} dM \right)^{\frac{1}{m-1}} \cdot (V(z, r))^{\frac{m-2}{m-1}},$$

for any  $z \in M$ , there exists a  $r(z) \in (0, R]$  such that

$$\int_{B_r(z)} |H|^{m-1} dM > \delta^{m-1} r(z).$$

Fix a point  $o \in M$ , we can find a ray  $\gamma : [0, +\infty) \rightarrow M$  parameterized by arclength. For any fixed  $R > 0$ ,

$$\gamma([0, R]) \subset \bigcup_{t \in [0, R]} B_{r(\gamma(t))}(\gamma(t)).$$

From a covering argument used in Theorem 1.1 of [To], we can find an at most countable sequence  $t_1, t_2, \dots, t_q, \dots \in [0, R]$  such that  $\sum_i r(\gamma(t_i)) \geq \frac{1}{4}R$  and when  $i \neq j$

$$B_{r(\gamma(t_i))}(\gamma(t_i)) \cap B_{r(\gamma(t_j))}(\gamma(t_j)) = \emptyset.$$

Then

$$\begin{aligned} \int_{B_{2R}(o)} |H|^{m-1} dM &\geq \sum_i \int_{B_{r(\gamma(t_i))}(\gamma(t_i))} |H|^{m-1} dM \\ &\geq \delta^{m-1} \sum_i r(\gamma(t_i)) \\ &\geq \delta^{m-1} \frac{1}{4}R. \end{aligned}$$



The result is proved. □

For complete surfaces in  $\mathbb{R}^n$  that satisfy the Gauss-Bonnet relation, we obtain the following result.

**Corollary 4.1** *Let  $\delta$  be as in theorem 4.1. If  $M$  is a complete noncompact surface in  $\mathbb{R}^n$  satisfying*

$$2\pi\chi(M) - \int_M K dM < 2\delta, \quad (22)$$

*where  $\chi(M)$  is the Euler characteristic of  $M$ , then*

$$\int_M |H| dM = +\infty.$$

**Proof.** From Theorem A of Shiohama [Sh], for any  $q \in M$ ,

$$\lim_{r \rightarrow \infty} \frac{2V(B_r(q))}{r^2} = 2\pi\chi(M) - \int_M K dM.$$

Observe that there is a misprint in the denominator of this expression in Shiohama's paper. So,

$$\lim_{r \rightarrow \infty} \frac{V(B_r(q))}{\pi r^2} < \delta.$$

Thus, Theorem 4.1 implies the result. □

**Remark 4.1** *The flat plane embedded in  $\mathbb{R}^n$  shows that the condition (22) is necessary.*

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